A THEOREM ON THE INSTABILITY OF EQUILIBRIUM

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We consider a holonomic system with stationary constraints and denote its kinetic energy and the force function by

$$T = \sum_{s, r=1}^{k} g_{sr} p_s p_r, \quad U = U(q_1, \ldots, q_k), \quad (U(0, \ldots, 0) = 0)$$

where $q_1, \ldots, q_k$ are the Lagrangian coordinates.

Theorem. The equilibrium is unstable, when the following conditions hold:

(1) a region A $(U > 0)$ for which the coordinate origin O is a point on the boundary, exists in the q_i space;

(2) a sphere B $(q_1^2 + \ldots + q_k^2 \leqslant \lambda)$ exists, whose radius is sufficiently small for the condition

$$f \equiv \sum_{i=1}^{k} \frac{\partial U}{\partial q_i} q_i \neq 0$$

to hold in the region $C = A \cap B$;

(3) the functions U and g_{sr} are continuously differentiable in C.

Proof. Regarding the canonical Hamilton equations as the equations of perturbed motion, we consider the function [1]

$$V = \sum_{i=1}^{k} q_i p_i$$

whose derivative, by virtue of these equations, has the form

$$V^{\cdot} = \sum_{s, r=1}^{k} \Big(2g_{sr} - \sum_{j=1}^{k} \frac{\partial g_{sr}}{\partial q_j} q_j\Big) p_s p_r + f$$

The positive definiteness of the quadratic form of the impulses in this expression has been shown, for sufficiently small numerical values of the coordinates q_i, in [2]. Using the Sylvester criterion, we obtain the radius of the corresponding neighborhood D of the coordinate origin of the q_i space.

We shall assume that the sphere B has been chosen to satisfy the condition $B \subset D$. According to the condition (3), the function f defined in C will have, at some point $M \in C$ the same sign as the derivative of U taken at the same point in the direction of the ray OM. But U is a function of a single variable in the direction of OM, vanishes at the point $L \in B$ of intersection of the ray OM with the boundary of A and is positive at the point M.

This, by virtue of the mean value theorem, implies the existence of a point $N \in LM$

at which $f > 0$. Then the condition (2) implies that the continuous function f conserves its sign in the region C. Consequently $V^{\cdot}$ is a positive function of the coordinates and impulses as long as the point representing the motion in the q_i space belongs to C. But this point cannot leave C by crossing the boundary of A if we put all $p_i = 0$ at the initial instant of time and choose the coordinates in C in accordance with the condition that $U_0 = \varepsilon$. As the system is conservative, the inequality $U \geqslant U_0$ holds.

The function f reaches its minimum positive value l on the compact $(U \geqslant \varepsilon) \cap B$. Equation

$$V = \int_{t_0}^{t} V^{\cdot} dt$$

implies that

$$V > l\,(t - t_0)$$

Consequently, a value l can be found for each $\varepsilon > 0$ such, that the inequality

$$\lambda < V = \sum_{i=1}^{k} p_i q_i \leqslant \frac{1}{2} \sum_{i=1}^{k} (p_i^2 + q_i^2)$$

is fulfilled not later than at the time

$$t = t_0 + \lambda / l$$

and this means instability in the Liapunov sense (*). Thus the classical methods [1, 2] appear to be applicable to more general problems.

Example 1. Let us consider a mathematical pendulum consisting of a weightless rod with a material point attached to it at one end. The other end of the rod is attached to a fixed point by means of a plane hinge. We denote by θ the angle of deflection of the rod from the vertical counted in the clockwise direction.

We assume that the pendulum is fitted with a spiral spring situated in the plane of oscillations. The inner end of the spring is rigidly fixed, while the outer end is connected to the pendulum by means of a catch in such a manner that the pendulum is disconnected from the spring when $\theta < 0$ and connected to it when $\theta \geqslant 0$. The moment developed by the spring is assumed to be equal to $M = -k^2\theta$.

At the position of equilibrium $\theta = 0$ considered here, the force function

$$U = \begin{cases} mgl\,(1 - \cos\theta) - {}^1/_2 k^2\theta^2, & \theta \geqslant 0 \\ mgl\,(1 - \cos\theta), & \theta < 0 \end{cases}$$

undergoes a second order discontinuity.

Example 2. Consider a rectilinear motion of a material point following the law

$$\theta^{\cdot\cdot} = \varphi(\theta) = \begin{cases} -\theta^2, & \theta < 0 \\ \theta \sin\theta^{-1}, & \theta \geqslant 0 \end{cases}$$

When $\theta < 0$, the force function

$$U = \int_0^{\theta} \varphi(\theta)\, d\theta$$

(*) After this note had gone to print, the author had learnt of a paper [3], in which the instability of equilibrium of a system with two degrees of freedom was proved for the case when the analytic force function has a minimum at the position of equilibrium.

satisfies, as in the previous example, the conditions formulated above which assert the instability (which is obvious in the present case) of equilibrium.

BIBLIOGRAPHY

1. Liapunov, A. M., General Problem of the Stability of Motion. M.-L., Gostekhizdat, 1950.
2. Chetaev, N. G., Stability of Motion. Papers on Analytical Mechanics. M., Izd. Akad. Nauk SSSR, 1962.
3. Hagedorn, P., Zur Umkehrung des Satzes von Lagrange über die Stabilität. ZAMP, Vol. 21, №5, 1970.

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ON THE OSCILLATIONS OF A SYSTEM OF COUPLED OSCILLATORS WITH ONE THIRD-ORDER RESONANCE

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The case when there is one resonance relation $\beta_1 = 2\beta_2$ between the frequencies of oscillators was studied in [1, 2]. We consider the possible case of a third-order resonance in the oscillations in a Hamiltonian system of nonlinearly coupled oscillators when there is one resonance relation of the form $\beta_1 + \beta_2 = \beta_3$ [1] between the frequencies of three oscillators. This problem was studied by using the method of secular perturbations in [6].

1. Statement of the problem. We consider a Hamiltonian system of nonlinearly coupled oscillators with the Hamiltonians

$$H(p, q) = H_2(p, q) + H_3(p, q) + \ldots + H_i(p, q) + \ldots \tag{1.1}$$

$$p = (p_1, \ldots, p_n), \quad q = (q_1, \ldots, q_n)$$

$$H_2(p, q) = \frac{1}{2} \sum_{\nu=1}^{n} \beta_\nu (q_\nu^2 + p_\nu^2) \qquad (\beta_\nu > 0) \tag{1.2}$$

Here $\pm i\beta_\nu$ are the eigenvalues of the linearized system; $H_i(p, q)$ are homogeneous polynomials of degree i. The quantities $\beta_\nu > 0$ corresponding to the frequencies of the "uncoupled" oscillators, i.e., to the case when all $H_i(p, q) = 0$ $(i \geqslant 3)$ in (1.1) are simply called frequencies in what follows.

Let there exist a relation

$$k_1\beta_1 + k_2\beta_2 + \ldots + k_n\beta_n = 0 \tag{1.3}$$

where the k_i are integers. Then we say that resonance occurs. The vector $k = (k_1, \ldots, k_n)$ is called the resonance vector, while the number $k = |k_1| + \ldots + |k_n|$ is called the order of the resonance. We consider a system of n oscillators in the case when there is only one linearly independent resonance relation (1.3) between the frequencies of the